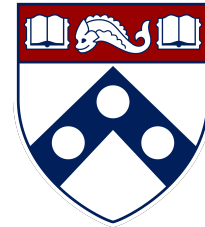
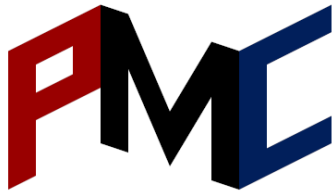


The University of Pennsylvania presents...

## The Fourth Annual Penn Math Contest! Intermediate Level, Individual Round

April 19th, 2025



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Welcome to the Penn Math Contest! We hope you enjoy your time here :)

- This exam contains 6 pages (including this cover page) and 20 questions. You have 90 minutes to answer as many questions correctly as possible. Every correct answer is worth four points, every blank answer is worth one point, and every wrong answer is worth zero points. The answer to every question is an integer between 0 and 999 (inclusive). Do NOT include units, percent signs, etc. in your answers.
- You are only allowed to use writing utensils, blank scratch paper, and (optionally) a ruler and compass. Do NOT use any devices that would give you an unfair advantage, such as calculators, computers, protractors, or phones.
- You are NOT allowed to consult outside sources such as the Internet or other people while you participate in the contest. The only exception is that for the team rounds, you are permitted to talk and collaborate, but only with members of your team.
- All questions are worth the same amount of points. If you get stuck on a question, feel free to skip it and move on to the next question.
- In the case of a tie, we will break the tie based on the first problem which one student got right and the other student got wrong.
- The top five individual students in the in-person contest will get medals!
- If there end up being cases which we haven't thought of and are not addressed by the rules, we will use our best common sense and judgment to sort the situation out. We will try to be as fair as possible.
- Good luck, you got this!

*Some brief notes about notation/terminology:*

- If you see brackets like this:  $\lfloor \ ]$ , then those are called “floor brackets” and they indicate rounding a number down to the nearest integer. For example:

$$\lfloor -4.4 \rfloor = -5, \quad \lfloor 18.83293 \rfloor = 18, \quad \lfloor 42 \rfloor = 42.$$

- If you see something like  $|x|$ , that is the absolute value symbol. If  $x \geq 0$ , then  $|x| = x$ , otherwise,  $|x| = -x$ .
- If you see “mod” such as  $15 \bmod 7$ , then that is short for “modulo” and it means to take the remainder after a division. For example, if you divide 15 by 7, you get an answer of 2 with a remainder of 1 left over, so

$$15 \bmod 7 = 1.$$

Similarly,  $5378 \bmod 1000 = 378$ .

- A range of numbers is “inclusive” if it includes the starting and ending number and “exclusive” if not. For example, “1 to 5 (inclusive)” means 1, 2, 3, 4, 5 while “1 to 5 (exclusive)” means 2, 3, 4.
- In general, if you have any questions about unfamiliar notation or terminology, feel free to ask one of the contest proctors.

*Questions:*

1. Say that we have a 3 inch by 6 inch rectangular piece of paper  $ABCD$ . Now, let’s apply the following instructions:
  1. Fold it the “hot dog way” so that the 6 inch sides line up. Unfold. This will create line  $\overline{EF}$  as the midsegment of the rectangle.
  2. From corner  $D$ , fold through that corner so that the nearest neighboring corner,  $A$ , lands on the fold  $\overline{EF}$  created in step 1. Call where the fold intersects the top corner point  $H$ , and call where point  $A$  lands on fold  $\overline{EF}$  point  $G$ . Note that  $AH = GH$ .
  3. Part of the 6 inch side,  $AH$  is now on the inside of the paper as segment  $GH$ . Fold the paper along line  $GH$ . This creates point  $I$  at the intersection with segment  $CD$ .  $G$ ,  $H$ , and  $I$  are now all on the same line.
  4. Unfold all of this.
  5. Cut along the folds in steps 2 and 3. This will give triangle  $DHI$ .

In the ideal case, this creates a polygon with area of  $a\sqrt{b}$  square inches in simplest form. What is  $10a + b \bmod 1000$ ?

2. From July 2, 1778 to July 20, 1778 both inclusive, the Second Continental Congress met at College Hall at the University of Pennsylvania (unfortunately not the same College Hall, but...). According to Wikipedia as of the time of writing this question, Penn had 24 members of the Continental Congress, while currently, we have 32 US Senators and 163 members of the US House of Representatives (and three US presidents). The ratio of the total number of Penn related Congress Members (Number of Senators + Number of Representatives) to the number of Penn related Continental Congress Members is  $a : b$  in simplest form. What is  $a + b$ ?

3. Penn built the ENIAC (Electronic Numerical Integrator and Computer) in 1945, making the first programmable, electronic, digital computer in the world. It cost \$487,000 (equivalent to \$6,900,000 in 2023). It is noted that taking a square root took  $13(d+1)$  cycles, where  $d$  is the number of digits in the result. So if we wanted to find the square root of the number 123456787654321, how many cycles would this take?
4. Cruise control was invented by Ralph Teetor, a Penn alum (1912 and 1930). Say that PASA is driving to Cherry Springs and is driving at a constant 70 miles per hour for 45 minutes. The distance to drive to Cherry Springs is about 250 miles. In this stretch of constant speed, the fraction of the distance did PASA drive is  $\frac{a}{b}$ . What is  $a + b$ ?
5. Floyd's Triangle is a triangle where the first row has one element, the second row has two elements, the third row has three elements, etc. These are then filled in row by row with natural numbers. So the first three rows look like this:

1  
2 3  
4 5 6

What is the sum of all of the numbers in the first 8 rows?

6. There have been a lot of rumors about how to access the second basement of the David Rittenhouse Laboratory. So far, only one person has been able to access it.

One rumor is that after finding a hidden door, you have to walk down a ramp to go down 9 feet in altitude, then drop down a 3 foot drop off. The slope of the ramp is  $1/10$ . If this had instead been a single ramp of a constant slope with no drop off, this would be shorter by a distance of  $a\sqrt{b} - c\sqrt{d} + e$  in simplified form? What is  $a + b + c + d + e$ ?

7. Sara is investigating the properties of what she calls prime paths. Her definition of a prime path is as follows: if  $a$  and  $b$  are positive integers, then a prime path from  $a$  to  $b$  is a list of positive integers where 1) the first number in the list is  $a$ , 2) the last number in the list is  $b$ , and 3) for every number in the list, it divided by the next number in the list (if there is one) is a prime number. For example, the list 60, 30, 10, 2 is a prime path from 60 to 2.

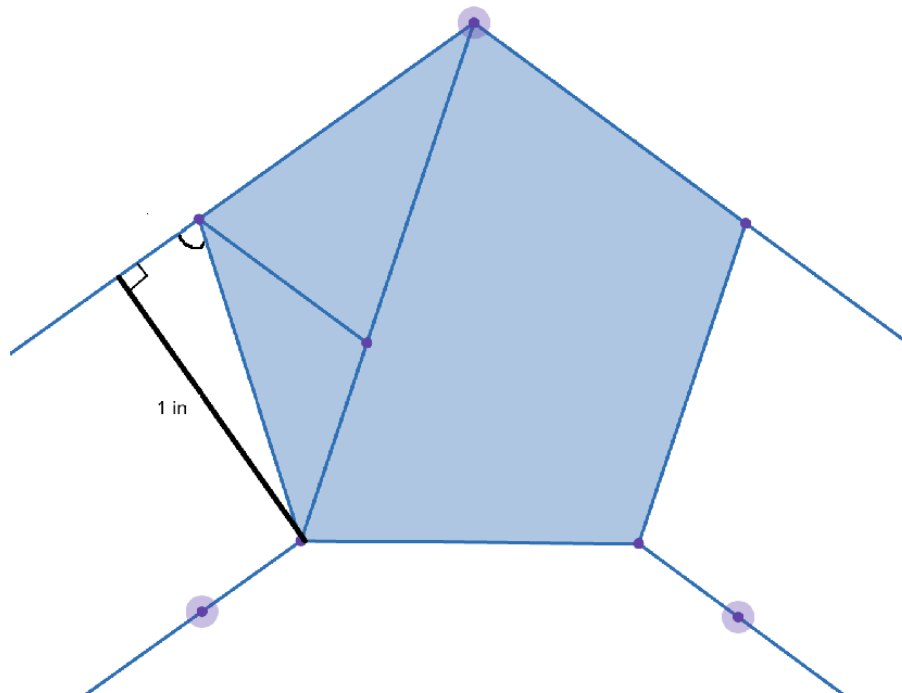
Sara wants to know how many prime paths there are from 2024 to 22. Can you help her find a path forward by answering that question?

8. Sophie is thinking of a number  $ABC$ , where  $A$ ,  $B$ , and  $C$  are all digits. She stumbles upon some interesting properties of this number:

- $A, B, C \geq 2$
- $\log_A B$  is rational
- $\log_B C$  is rational
- $Ax^2 + Bx + C$  has only rational roots
- For every prime factor  $p$  of  $ABC$ ,  $ABC$  is also divisible by  $p^2$ .
- The sum of the distinct prime factors of  $ABC$  is greater than 10.

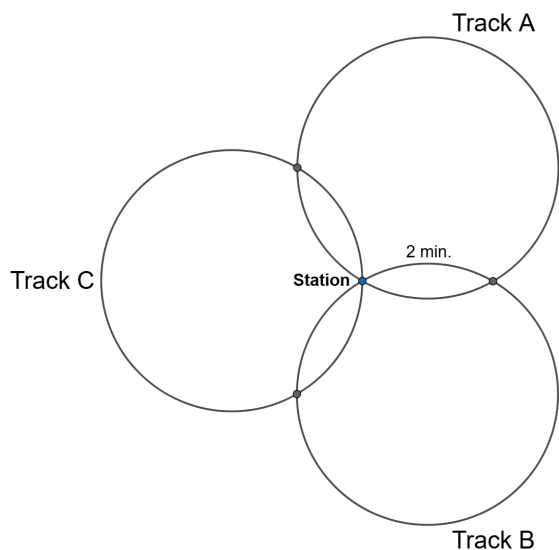
What number is Sophie thinking of?

9. Triangle  $ABC$  has right angle  $C$ ,  $BC = 21$  and  $AB = 29$ . Let  $D$  be the point on  $AB$  between  $A$  and  $B$  such that the areas of triangles  $ACD$  and  $BCD$  are equal, and let  $E$  be the point on  $AB$  between  $A$  and  $B$  such that the perimeters of triangles  $ACE$  and  $BCE$  are equal. The area of triangle  $CDE$  is then  $\frac{a}{b}$  in simplest form. What is  $a + b$ ? 29  
21  
20
10. Consider a regular tetrahedron with sidelength 1. Place a sphere of radius  $r$  on each vertex. Let the smallest  $r$  such every point in the tetrahedron is contained in at least one of the spheres be  $\frac{\sqrt{a}}{b}$ . What is  $a + b$ ? 134
11. If  $S$  and  $T$  are two sets (collections of objects), then a relation from  $S$  to  $T$  is a set of ordered pairs of the form  $(s, t)$ , where  $s$  is an element of  $S$  and  $t$  is an element of  $T$ . That relation is a function if it pairs every element of  $S$  with exactly one element of  $T$ .
- Rory decides to randomly construct a relation  $R$  by the following procedure: For each possible ordered pair of an element of  $S$  and an element of  $T$ , Rory flips a coin. If it lands heads (50% probability), he adds the ordered pair to  $R$ . Otherwise, he doesn't add it to  $R$ . If  $S$  has 3 elements and  $T$  has 8 elements, then the probability that  $R$  ends up being a function can be written as  $2^{-n}$ . What is  $n$ ?
12. You roll an 8 sided die until you see a 5. The expected value of the largest value you roll can be represented as a fraction  $\frac{a}{b}$ , in simplest terms. What is  $a + b$ ?
13. David has a long strip of paper which has a width of 1 inch. He ties it into an overhand knot, and the knot forms a regular pentagon. Then, the perimeter of the regular pentagon can be written as  $\frac{a}{\sin(b^\circ)}$ , in inches, where  $a$  and  $b$  are positive integers and  $b \leq 90$ . What is  $100a + b$ ?



14. During PMC II, Penn Math Contest was trying to balance graphic design and social media posts to try to market the contest. Let  $x$  be the units of graphic design and  $y$  be the units of social media posts. Each unit of graphic design requires 3 units of work, while each unit of social media posts requires 2 units of work. PMC had a total of 83 units of work. If the quality of marketing measured in Klein Bottles is  $12 \left( \frac{2xy+4x+6y+12}{3x+2y+13} \right)$ , then what is the maximum possible quality of PMC's marketing?
15. You are locked in the DRL basement (floor 0) and you want to get to the 4th floor. You walk around until you find a staircase. On the basement the staircase is guaranteed to go up. However, on the 1st, 2nd, and 3rd floor the staircase has a 50% chance of going up and a 50% chance of going down. What is the expected number of flights of stairs you will take?
16. Tempos are measured in beats per minute. How many tempos between 60 and 180 both inclusive (the range of tempos that are commonly used in UPGRADE games) have the property that 8 measures in  $4/4$  (4 beats per measure and the quarter note gets the beat) takes an integer number of seconds?
17. David and Nihar are putting together the PMC IV test. They have 2025 questions to review. Each day exactly one of them can review up to 50 questions. However, they must review at least 30 so that Darren doesn't call them lazy. Suppose that David goes first on day 1, Nihar on day 2, ... . They continue until all questions have been reviewed. Whoever reviews the last question will go to Darren and claim they did all the work. To reward them, Darren will buy that person dinner from HipCityVeg. Nihar has the option to pay Eric to secretly delete some questions from the pool at a rate of 10 cents per question. Assume both David and Nihar play optimally, and let  $c$  denote the minimum number of cents that Nihar can pay Eric and guarantee the reward from Darren. What is  $c + 1$ ?
18. You have 6 urns. In all urns, there are 10 blue balls and 7 red balls. You randomly draw a ball from urn 1 and place it in urn 2. Then you randomly draw a ball from urn 2 and place it in urn 3. You continue this process until you draw a ball from the last urn. The probability that you draw a blue ball can be represented by  $\frac{a}{b}$ , a fraction in simplest terms. What is  $a + b$ ?
19. There are 3 trains A, B, and C which all travel clockwise along their own separate, identical, circular tracks as shown below. All trains begin stopped at the station in the central intersection of the three circular tracks. Each train travels at the same fixed speed. Train A will start along Track A at time  $t = 0$  minutes, and Train B will come next on Track B, followed by Train C on Track C. For any train, the length of the track from the station to the first intersection point of two tracks takes exactly two minutes to traverse.

How many distinct pairs of positive integers  $(t_1, t_2)$  are there such that if the conductor starts Train B at time  $t_1$  and Train C at time  $t_2$ , all three trains can continue indefinitely and never collide? Assume each train has zero length, taking up a single point in space, and a collision occurs when two trains, at least one of which is moving, occupy the same point in space at the same time.



20. We will say a point  $p$  in 2-dimensional space is inside a parabola if we can construct a line through  $p$  that intersects the parabola at two different points such that  $p$  is in between the two points of intersection.

We construct a set  $S$  by adding all points  $(x, y)$  which have probability  $\frac{1}{6}$  of being on or inside a parabola selected uniformly at random from those with vertex at  $(0, 0)$  and a focus at a distance of one unit from the origin. The set  $S$  forms a circle with radius  $a\sqrt{b}$ , when expressed in simplest radical form. Find  $a^2 \cdot b$ .